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Making non-associative algebra associative

Pei-Ming Ho

Department of Physics, National Taiwan University
Taipei 106, Taiwan, R.O.C.
E-mail: pmho@phys.ntu.edu.tw

Abstract: Based on results about open string correlation functions, a nonassociative algebra was proposed in a recent paper for D-branes in a background with nonvanishing H. We show that our associative algebra obtained by quantizing the endpoints of an open string in an earlier work can also be used to reproduce the same correlation functions. The novelty of this algebra is that functions on the D-brane do not form a closed algebra. This poses a problem to define gauge transformations on such noncommutative spaces. We propose a resolution by generalizing the description of gauge transformations which naturally involves global symmetries. This can be understood in the context of matrix theory.

Keywords: D-branes, Non-Commutative Geometry.

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Contents

1. Introduction

In an interesting paper of Cornalba and Schiappa [1], they calculated the n-point functions for open strings ending on a D-brane in a NS-NS B field background with $H \neq 0$. From the correlation functions they tried to extract information about the algebra of functions on the D-brane worldvolume. They found that Kontsevich's formal expression for the $*$-product can be used to reproduce the correlation functions. However, this product is nonassociative because B is not symplectic when $H \neq 0$.

On the other hand, in an earlier paper [2], we derived an associative algebra for the D-brane worldvolume by quantizing open strings ending on a D-brane in curved space with a nontrivial B field background. The question is whether our algebra can also reproduce the correlation functions.

When one tries to extract the algebra of functions on the D-brane from the correlation functions, the answer is not unique, because the correlation functions are just numbers, which we want to interpret as the integrals of functions on a noncommutative space. Since we only have information about the integrals of functions, instead of the functions themselves, we can not directly obtain the algebra without ambiguity. It is possible to have many different algebras that reproduce the same correlation functions after integration.

On the other hand, if we try to derive the algebra of functions by quantizing an open string on the D-brane, the result is always an associative algebra. We should just interpret the algebra of the endpoint coordinates X and momentum P as the algebra of functions and derivatives on the D-brane.

It is well known that for a constant B field background, quantization of open string coordinates [3] and calculation of correlation functions [4] give the same noncommutativity of D-brane worldvolume. It would be nice to have this kind of agreement for a generic background.

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In this paper, we show that the algebra of [2] can also reproduce the open string n-point functions, but it has the merit of being associative. The novel property of this algebra is that the algebra of functions and the algebra of derivatives are mixed up. The functions do not form a closed algebra by themselves. This makes it hard to formulate a gauge theory on such noncommutative spaces. In the end of this paper we propose a way to generalize the notion of gauge transformations for such noncommutative spaces. It naturally includes a description of global symmetries, and is reminiscent of the situation in matrix compactifications.

2. The non-associative algebra

It is well known that the noncommutative algebra

$$[x^a, x^b] = i\theta^{ab} \tag{2.1}$$

for a constant anti-symmetric tensor θ can be realized on classical commutative functions by the $*$-product

$$f * g = fg + \frac{i}{2}\theta^{ab}(\partial_a f)(\partial_b g) - \frac{1}{8}\theta^{ac}\theta^{bd}(\partial_a\partial_b f)(\partial_c\partial_d g) + \mathcal{O}(\theta^3)\,. \tag{2.2}$$

For a generic Poisson structure $\tilde{\theta}(x)$, the Kontsevich formula [5] gives

$$\begin{aligned} f \bullet g = {}& fg + \frac{i}{2}\tilde{\theta}^{ab}(\partial_a f)(\partial_b g) - \frac{1}{8}\tilde{\theta}^{ac}\tilde{\theta}^{bd}(\partial_a\partial_b f)(\partial_c\partial_d g) - \\ & - \frac{1}{12}\tilde{\theta}^{ad}(\partial_d\tilde{\theta}^{bc})\left((\partial_a\partial_b f)(\partial_c g) - (\partial_b f)(\partial_a\partial_c g)\right) + \mathcal{O}(\tilde{\theta}^3)\,. \end{aligned} \tag{2.3}$$

When the field strength

$$H_{abc} = (\partial_a \tilde{B}_{bc}) + (\partial_b \tilde{B}_{ca}) + (\partial_c \tilde{B}_{ab}) \tag{2.4}$$

for the NS-NS B field background $\tilde{B}$ vanishes, $\tilde{B}$ defines a symplectic structure on the D-brane and its inverse gives the Poisson structure

$$\tilde{\theta} = \tilde{B}^{-1} \tag{2.5}$$

which defines via (2.3) the noncommutativity of the D-brane worldvolume in the zero slope limit of Seiberg and Witten [6].

For a more general matrix of functions $\tilde{\theta}$ which is not a Poisson structure, correpsonding to the case $H \neq 0$, the algebra defined by the Kontsevich formula is not associative. The nonassociativity is

$$(f \bullet g) \bullet h - f \bullet (g \bullet h) = \frac{1}{6}K^{abc}(\partial_a f)(\partial_b g)(\partial_c h) + \cdots\,, \tag{2.6}$$

where

$$K^{abc} = \tilde{\theta}^{ad}(\partial_d \tilde{\theta}^{bc}) = \tilde{\theta}^{ad}\tilde{\theta}^{be}\tilde{\theta}^{cf} H_{def} \,. \tag{2.7}$$

Throughout this paper we will only keep terms up to the first order of K and to the 2nd order of $\tilde{\theta}$.

For simplicity, let us consider the case

$$\tilde{B}_{ab} = B_{ab} + \frac{1}{3} H_{abc} x^c + \cdots \,, \tag{2.8}$$

where B and H are constant anti-symmetric tensors. This is the same case considered in [1]. We refer to [1] for discussions on the effect of nonzero H on the curvature, as well as many of our conventions and notations.

The last term in the Kontsevich formula (2.3) vanishes for the choice (2.8) because its coefficient is proportional to the anti-symmetric tensor K^{abc}, whose indices are contracted with derivatives on f or g. The formula (2.3) is simplified to

$$\begin{aligned} f \bullet g &= fg + \frac{i}{2}\tilde{\theta}^{ab}(\partial_a f)(\partial_b g) - \frac{1}{8}\tilde{\theta}^{ac}\tilde{\theta}^{bd}(\partial_a\partial_b f)(\partial_c\partial_d g) + \cdots \\ &= f * g - \frac{i}{6} K^{abc} y_c * (\partial_a f) * (\partial_b g) + \cdots \,, \end{aligned} \tag{2.9}$$

where

$$y_a = B_{ab} x^b \,, \tag{2.10}$$

and the $*$-product is defined by (2.2) for $\theta = B^{-1}$. In deriving (2.9) we used the relation

$$\tilde{\theta}^{ab} = \theta^{ab} - \frac{1}{3} K^{abc} y_c + \cdots \tag{2.11}$$

which follows from (2.5) and (2.8). Note that the last term in (2.9) can also be written as

$$-\frac{i}{6} K^{abc}(\partial_a f) * (\partial_b g) * y_c \tag{2.12}$$

because

$$\begin{aligned} y_a * f &= y_a f + \frac{i}{2}(\partial_a f) \,, \\ f * y_a &= y_a f - \frac{i}{2}(\partial_a f) \,, \end{aligned}$$

and K is totally antisymmetric.

From (2.9) it is straightforward to calculate

$$(\cdots((f_1 \bullet f_2) \bullet f_3) \cdots \bullet f_n) = f_1 * f_2 * \cdots * f_n + \sum_{i<j} V_{ij} \,, \tag{2.13}$$

where

$$V_{ij} = -\frac{i}{6} K^{abc} y_c * f_1 * \cdots * (\partial_a f_i) * \cdots * (\partial_b f_j) * \cdots * f_n \,. \tag{2.14}$$

Similarly,

$$f_1 \bullet (f_2 \bullet \cdots (f_{n-1} \bullet f_n) \cdots) = f_1 * f_2 * \cdots * f_n + \sum_{i<j} V'_{ij}\,, \tag{2.15}$$

where

$$V'_{ij} = -\frac{i}{6} K^{abc} f_1 * \cdots * (\partial_a f_i) * \cdots * (\partial_b f_j) * \cdots * f_n * y_c\,. \tag{2.16}$$

It was shown in [1] that for open strings ending on a D-brane in this B field background, the 2-point function is given by

$$\int f \bullet g + \int \frac{1}{3} B_{bc} K^{abc} y_a * f * g + \cdots\,. \tag{2.17}$$

The last term is the contribution from contracting two fields in the interaction term of the open string action, and it has the same form

$$\int \frac{1}{3} B_{bc} K^{abc} y_a * f_1 * f_2 * \cdots f_n \tag{2.18}$$

for n-point functions. They can be taken care of by a modification of the measure for all correlation functions [1]

$$\int F \to \int (1 + \frac{1}{3} B_{bc} K^{abc} y_a) * F\,, \tag{2.19}$$

so we will ignore such terms from now on.

The 3-point correlation function (up to a term of the form (2.18)) is reproduced by the integral

$$\int (f \bullet g) \bullet h = \int f \bullet (g \bullet h)\,. \tag{2.20}$$

The nonassociativity of the $\bullet$-product does not affect the 3-point function.

More generally, we have

$$\begin{aligned} \int (\cdots (f_1 \bullet f_2) \cdots \bullet f_n) &= \int (f_1 \bullet \cdots (f_{n-1} \bullet f_n) \cdots) \\ &= \int (f_1 * \cdots * f_n + \sum_{i<j} V_{ij})\,. \end{aligned} \tag{2.21}$$

However, for n-point functions with $n > 3$ one has to use a linear combination of the $\bullet$-products with various orderings, weighed by coefficients depending on the modules.

3. The associative algebra

In [2], the symplectic structure for the endpoint coordinates of an open string ending on a D-brane in a background with nonvanishing H is derived in the low energy limit.

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Inverting the sympletic two-form for X and $(\partial_\sigma X)$, we find the Poisson brackets for the coordinates x at $\sigma = 0$ and the momentum density at $\sigma = 0$ times π (which is the same as the total momentum in our approximation)

$$p_a = \pi \tilde{\theta}^{-1}_{ab} X'^b + \cdots \tag{3.1}$$

as

$$(x^a, x^b) = \tilde{\theta}^{ab} - \frac{1}{3} K^{abc} p_c + \cdots , \tag{3.2}$$

$$(x^a, p_b) = \delta^a_b + \frac{1}{6} \tilde{\theta}^{-1}_{bc} K^{acd} p_d + \cdots , \tag{3.3}$$

$$(p_a, p_b) = 0 + \cdots , \tag{3.4}$$

where we denote $\hat{\mathcal{F}}^{-1}$ by $\tilde{\theta}$ and keep only terms up to the first order in K. (K is defined in (2.7).) One can check that all Jacobi identities are satisfied up to our approximation.

A peculiar property of this algebra is that the commutator of two functions of x is not a function of x only, but a function of both x and p. Another interesting charater of this algebra is that it is impossible to realize p by a function f of x such that $(p, x) = (f, x)$. That is, the derivative p is not an "inner derivative". These properties are the algebraic manifestations of the fact that $H \neq 0$.

Poisson brackets turn into commutation relations upon quantization

$$[f, g] = i(f, g) + \cdots . \tag{3.5}$$

The idea about $*$-product is that the quantum algebra can be realized on commutative functions by defining a new product

$$f \diamond g = fg + \frac{i}{2}(f, g) + \cdots . \tag{3.6}$$

If $F = 0$, $\hat{\mathcal{F}}$ is just the B field background in the Seiberg-Witten limit [6]. Consider the case $\hat{\mathcal{F}} = \tilde{B}$ given by (2.8) in the previous section. We find that (3.2) differs from what we get from (2.3) by the term linear in p. Thus we should define a new product

$$f \diamond g = f \bullet g - \frac{i}{6} K^{abc} (\partial_a f)(\partial_b g) p_c + \cdots \tag{3.7}$$

to account for the last term in (3.2). This new term that modifies the $\bullet$-product is precisely what is needed to make it associative

$$(f \diamond g) \diamond h = f \diamond (g \diamond h) . \tag{3.8}$$

Note that here we choose to use p as a c-number, and p is a derivative in the sense that $x^a \diamond p_b - p_b \diamond x^a \simeq i\delta^a_b + \cdots$.

Now we have to check that this new product generates the same 2-point function after integration. But how do we integrate the terms involving derivatives? Consider the space of all functions of x as the Hilbert space of a one-particle system. Any state in the Hilbert space can be obtained by acting a function of x on the "vacuum" $\rangle$, which corresponds to the constant function. The action of p on a state is determined by the commutation relations between p and x, in addition to the action of p on $\rangle$ give by

$$p_a\rangle = 0\,. \tag{3.9}$$

Now we define the integration of $f(x,p)$ by $\langle f(x,p)\rangle$, where $\langle$ is the hermitian conjugate of $\rangle$. In our notation, the inner product $\langle f|g\rangle = \langle f^\dagger g\rangle$, and so $\langle f(x)\rangle = \langle 1|f(x)\rangle$, which is just the integration of $f(x)$ when $\tilde{\theta} = 0$.

The translational invariance of the integration is guaranteed by (3.9). To calculate the integral, one simply commutes p to the right to annihilate $\rangle$, so that it turns into an integration of a pure function of x. This kind of definition for integrals on noncommutative spaces has been used in many occasions [7].

One can check that

$$\langle f \diamond g\rangle = \int f \bullet g\,, \tag{3.10}$$

and so the new product is consistent with the 2-point function.

For the 3-point function, we find

$$f\diamond g\diamond h = f\bullet(g\bullet h) - \frac{i}{6}K^{abc}[(\partial_a f)(\partial_b g)h + (\partial_a f)g(\partial_b h) + f(\partial_a g)(\partial_b h)]p_c + \cdots\,, \tag{3.11}$$

and so

$$\langle f\diamond g\diamond h\rangle = \int f\bullet(g\bullet h)\,. \tag{3.12}$$

Thus we check that the new $\diamond$-product also reproduces the 3-point function correctly. What we gained by the modification to the $\bullet$-product is that our algebra is now associative.

Denote the insertion coordinates at the boundary of the open string by τ_i. The 4-point function depends on the module m which is the cross ratio

$$m = (\tau_4-\tau_3)(\tau_2-\tau_1)(\tau_4-\tau_2)^{-1}(\tau_3-\tau_1)^{-1}\,. \tag{3.13}$$

The integral

$$\langle f_1\diamond\cdots\diamond f_4\rangle = \int\left(f_1 * \cdots * f_4 + \sum_{i<j} V_{ij}\right) \tag{3.14}$$

properly reproduces the 4-point function [1] except the term depending on the module

$$\frac{1}{6}L(m)K^{abc}\int f_1(\partial_a f_2)(\partial_b f_3)(\partial_c f_4)\,, \tag{3.15}$$

where $L(m)$ is a function ranging between 0 and 1 [1]. One should integrate over m to make connection with the D-brane action. This term can be taken care of separately by a term of the form

$$K^{abc} \int f_1 \diamond (\partial_a f_2) \diamond (\partial_b f_3) \diamond (\partial_c f_4) \,. \tag{3.16}$$

In general,

$$\langle f_1 \diamond \cdots \diamond f_n \rangle = \int (f_1 * \cdots * f_n + \sum_{i<j} V_{ij}) \,. \tag{3.17}$$

To take care of the module dependent terms of the n-point functions, one has to superpose $\bullet$-products of various orderings [1]. For the $\diamond$-product, one has to take care of these terms separately. Since we do not have to use more $\diamond$-products to write down an n-point function, the $\diamond$-product is as good as the $\bullet$-product for the purpose of generating correlation functions. It is also possible that we do not have to worry about the 4-point functions because the quartic terms in the low energy D-brane action may be uniquely determined by gauge symmetry after the quadratic and cubic terms are given. The benefit of using the $\diamond$-product is of course that it is associative, and so one can use it to formulate algebraic structures of the theory such as symmetries.

4. Noncommutative gauge theory

In this section we discuss how to construct a gauge theory on the noncommutative space defined by (3.2)–(3.4). This noncommutative space is quite different from the cases with $H = 0$ in that the functions of x do not form a closed subalgebra. When one makes a gauge transformation on a field $\phi(x)$ in the adjoint representation

$$\phi \to \phi' = U \diamond \phi \diamond U^\dagger \,, \tag{4.1}$$

the result ϕ' is generically not a function of x although ϕ and U are functions of x only. This poses a serious problem in constructing a gauge field theory.

We propose that the natural notion of gauge transformations here is to restrict U to be functions of x and p such that ϕ' will be a function of x for any ϕ. From the viewpoint of the matrix model, both x and p have the same origin as (infinite dimensional) matrices. Note that in the classical case this notion of gauge symmetry does not completely coincide with the usual definition. We shall further restrict the symmetry so that it has the correct classical limit when $\tilde{\theta} \to 0$. In fact, the notion of gauge symmetry has already been generalized for compactified matrix theory [8, 9, 10]. It has been pointed out that both global symmetries and gauge symmetries for the theory after compactification come from gauge symmetries of the original matrix theory [9, 10]. For example, the quotient condition for the matrix model compactified on a torus [11] is

$$U_j^\dagger X_i U_j = X_i + 2\pi\delta_{ij}R_j\,. \tag{4.2}$$

For the dual theory in which X_i is identified with the covariant derivative,

$$X_i \to u^\dagger X_i u \tag{4.3}$$

is a gauge transformation if $uU_i = U_i u$. It is a global (translational) symmetry if $uU_i = q_i U_i u$ for some phase factor q_i [9]. The symmetries of the compactified theory are given by any unitary transformation that commutes with the quotient conditions. Our proposal is in the same spirit.

For a field $\phi(x)$ in the adjoint representation, an infinitesimal gauge transformation by λ is

$$\delta\phi(x) = [\lambda, \phi(x)]\,. \tag{4.4}$$

For λ to make a valid transformation, we require that $\delta\phi$ is a function of x for any function $\phi(x)$. Expanding $\lambda(x,p)$ in powers of p

$$\lambda = \lambda_0(x) + \lambda_1^a(x)p_a + \lambda_2^{ab}(x)p_a p_b + \cdots\,, \tag{4.5}$$

where $\lambda_2^{ab} = \lambda_2^{ba}$, we find that the requirement is matched if

$$\begin{aligned}\lambda_1^a &= -2\theta^{ab}(\partial_b\lambda_0) + \theta^{ab}\zeta_b\,,\\ \lambda_2^{ab} &= \theta^{ac}\theta^{bd}((\partial_c\partial_d\lambda_0) + \xi_{cd})\,,\end{aligned} \tag{4.6}$$

where ζ satisfies

$$(\partial_a\zeta_b) - (\partial_b\zeta_a) = \frac{1}{3}H_{abc}\theta^{cd}\zeta_d\,, \tag{4.7}$$

and ξ is defined by

$$\xi_{ab} = -\frac{1}{4}((\partial_a\zeta_b) + (\partial_b\zeta_a))\,. \tag{4.8}$$

It follows from this that

$$\delta\phi = -i\theta^{ab}(\partial_a\lambda_0)(\partial_b\phi) + i\theta^{ab}\zeta_a(\partial_b\phi) + \cdots \tag{4.9}$$

is a function of x only.

It is interesting to note that ζ defines the Killing vector for $\tilde{\theta}$. Eq. (4.7) is precisely the constraint for the coordinate transformations

$$\delta x^a = \theta^{ab}\zeta_b \tag{4.10}$$

which keep $\tilde{\theta}$ invariant at $x = 0$, and the contribution of ζ to $\delta\phi$ in (4.9) is exactly this coordinate transformation. This means that the degrees of freedom in ζ

correspond to global symmetries, and those in $\lambda(x)$ correspond to gauge transformations. As we mentioned earlier, this is in close analogy with the situation in matrix compactifications.

For the gauge potential A, we can define its gauge transformation via the covariant derivative. If we define the covariant derivative by $D = p + A(x)$, we will find that it is not of the same form after a gauge transformation. One simple way out is to define it by $D = x + A$, which appears naturally in the context of matrix model [12, 6, 13]. Requiring D to transform in the adjoint representation determines the gauge transformation of A

$$\delta A^a = [\lambda, D^a]\,. \tag{4.11}$$

For a field $\psi(x)$ in the fundamental representation, we can define its gauge transformation ($\zeta = 0$) for $\psi\rangle$ as

$$\begin{aligned}\delta\psi\rangle &= \lambda \diamond \psi\rangle \\ &= \left(\lambda_0\psi - \frac{i}{2}\theta^{ab}(\partial_a\lambda_0)(\partial_b\psi)\right)\rangle + \cdots\,.\end{aligned} \tag{4.12}$$

If we consider $\lambda \diamond \psi$ without acting on $\rangle$, there will also be terms depending on p. Thus when trying to write down a gauge invariant action, we can have terms like $\langle\psi^\dagger\psi\rangle$, but not terms like $\langle(\psi^\dagger\psi)^2\rangle$.

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